Composite Functions

1. Let $z=f(x,y)$ be a function of two variables x and y.

a) Further x is a function of u and y is also function of u then $z=f(u)$ is a composite function of u.

Eg. $z=\sin(x+y)$, $x=u^2$, $y=2u \Rightarrow z=\sin(u^2+2u)$ is a composite function of u.

b) Further if x and y both are functions of u and v then $z=f(u,v)$ is a composite function of u and v

Eg. $z=\sin(x+y)$; $x=u\cdot v$ $y=u+v$. then $z=\sin(uv+u+v)$ is a composite function of u & v.

Differentiation of composite functions

i) If $z=f(x,y)$ and further x and y are function of t then z is a composite function of t.

Formula 1

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

→ Derivative of z w.r.t t.

2) If $z=f(x,y)$ and further x and y are functions of u and v then z is a composite function of u and v. Also,

Formula 2

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

→ Partial Derivatives of z wrt u and v respectively.

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2. Implicit and Explicit Functions

A relation of the form $f(x,y)=c$ in which x and y cannot be seperated out, is called implicit function, otherwise it is called explicit function

Eg: $x^2+y^2-2xy=0$ is an implicit function

$y^2=(x^2+2)$ is an example of Explicit function

Q.1 If $z=x^2-y^2$

$x=e^t\cos t \quad y=e^t\sin t$.

find $\frac{dz}{dt}$.

$z=z(x,y)$
$x=x(t)$
$y=y(t)$
$\Rightarrow z=z(t)$.

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=2x \quad \frac{\partial z}{\partial y}=-2y\\ \frac{dx}{dt}&=e^t(-\sin t)+\cos t\, e^t\\ \frac{dy}{dt}&=e^t(\cos t)+\sin t\, e^t\end{aligned}\right\}\ (1)$$

Put (1) in $\frac{dz}{dt}$, we get

$$\frac{dz}{dt}=2x(-e^t\sin t+e^t\cos t)-2y(e^t\cos t+e^t\sin t)$$

$$=(2x-2y)e^t\cos t-(2x+2y)e^t\sin t$$

Put $x=e^t\cos t \quad y=e^t\sin t$

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$= 2(e^t\cos t - e^t\sin t)\,e^t\cos t - 2(e^t\cos t + e^t\sin t)\,e^t\sin t$ ②

$= 2e^{2t}(\cos^2 t - \sin t\cos t) - 2e^{2t}(\cos t\sin t - \sin^2 t)$

$= 2e^{2t}(\cos^2 t - \sin^2 t) - 2e^{2t}(2\sin t\cos t)$

$= 2e^{2t}(\cos 2t) - 2e^{2t}(\sin 2t)$

$= 2e^{2t}(\cos 2t - \sin 2t)$

$\because \sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

2. If $u = f(r,s)$ $r = x+y$, $s = x-y$

Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Proof Apply formula ②.

$u = f(r,s)$, $r = r(x,y)$, $s = s(x,y)$ $\Rightarrow$ u is a composite function of x & y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot 1 + \frac{\partial u}{\partial s}\cdot 1 \quad -①$$

$\left(\because \frac{\partial r}{\partial x} = 1 \;\; \frac{\partial s}{\partial x} = 1\right)$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}\,1\cdot + \frac{\partial u}{\partial s}(-1) \quad -②$$

Add ① and ②

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} = 2\frac{\partial u}{\partial r}$$

Hence proved

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③ If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then show that

$$x^2 \cdot \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z} = 0.$$

Proof Given $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$.

Let $V = \frac{y-x}{xy}$, $W = \frac{z-x}{xz}$

$\Rightarrow u = f(V, W)$.

Also $V = V(x, y, z)$, $W = W(x, y, z)$

$\Rightarrow$ u is a composite function of x, y, z.

Apply Formula ②

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial V} \cdot \frac{\partial V}{\partial x} + \frac{\partial u}{\partial W} \frac{\partial W}{\partial x}.$$

$$V = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$$

$$W = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}.$$

$$\frac{\partial V}{\partial x} = \frac{-1}{x^2} \qquad \frac{\partial V}{\partial y} = \frac{1}{y^2}$$

$$\frac{\partial V}{\partial z} = 0.$$

$$\frac{\partial W}{\partial x} = \frac{-1}{x^2} \qquad \frac{\partial W}{\partial z} = \frac{1}{z^2}$$

$$\frac{\partial W}{\partial y} = 0.$$

Put these values in $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial V}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial W}\left(\frac{-1}{x^2}\right). \quad \text{— ①}$$

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$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}\cdot 0 \quad = \quad \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$= \frac{\partial u}{\partial v}\cdot 0 + \frac{\partial u}{\partial w}\cdot\left(\frac{1}{z^2}\right)$$

Now

$$x^2\cdot\frac{\partial u}{\partial x} + y^2\cdot\frac{\partial u}{\partial y} + z^2\cdot\frac{\partial u}{\partial z} = x^2\left[\frac{\partial u}{\partial v}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial w}\left(\frac{-1}{x^2}\right)\right]$$

$$+ y^2\left[\frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right)\right] + z^2\left[\frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right)\right]$$

$$= -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$= 0.$$

Hence proved.

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